Lecture 7 §1.11 1.12

Ex 1: $8x^3 y\,dx = (x^4+1)\,dy$ $y(0)=2$ separable

$\frac{8x^3}{x^4+1}\,dx = \frac{1}{y}\,dy$ Let $u = x^4+1$

$\int \frac{2\,du}{u} = \ln|y| + C$

$2\ln|u| = 2\ln(x^4+1) = \ln|y| + C$

$y = C(x^4+1)^2$

$y(0) = C = 2 \Rightarrow y = 2(x^4+1)^2$

Ex 2: $(y\cos xy - \sin x)\,dx + (x\cos xy)\,dy = 0$ Exact.

$\Phi(x,y) = \int (y\cos xy - \sin x)\,dx = \sin(xy) + \cos x + h(y)$

$\frac{\partial}{\partial y}\Phi = x\cos xy + h'(y) = x\cos xy \Rightarrow h'(y) = 0 \Rightarrow h(y) = C$

$\Phi(x,y) = \sin(xy) + \cos x + C = 0$

Ex 3: $y' - \frac{1}{2x}y = x$ $0 < x$, $y(1) = \frac{1}{3}$ $\Leftrightarrow$ The original problem is linear 1st order DE

$I(x) = e^{\int -\frac{1}{2x}dx} = e^{\ln\frac{1}{\sqrt{x}}} = \frac{1}{\sqrt{x}}$

$\left(\frac{1}{\sqrt{x}}y\right)' = x \cdot \frac{1}{\sqrt{x}} = \sqrt{x} \Rightarrow \frac{1}{\sqrt{x}}y = \int \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}} + C.$

$y = \frac{2}{3}x^2 + C\sqrt{x}$

$y(1) = \frac{2}{3} + C = \frac{1}{3}$ $C = -\frac{1}{3}$

EX 4: $y'' = \frac{2}{1-y}(y')^2$ 2nd order DE with $f = f(y, y')$

Let $u = y'$ $\frac{du}{dx} = y'' = u\frac{du}{dy}$

$$u\frac{du}{dy} = \frac{2}{1-y}u^2$$

We can divide both sides by u

But this excludes a particular solution $u=0$

$\Updownarrow$

$\boxed{y = C}$

$$\Downarrow$$

$$\frac{du}{dy} = \frac{2}{1-y}u \quad \text{separable}$$

$$\Downarrow$$

$$\int\frac{du}{u} = \int\frac{2}{1-y}dy + C_1 \Rightarrow \ln|u| = -\ln(1-y)^2 + C_1$$

$$\Rightarrow u = C_1\frac{1}{(1-y)^2} \Rightarrow \frac{dy}{dx} = C_1\frac{1}{(1-y)^2} \text{ : separable}$$

$$(1-y)^2dy = C_1\,dx \Rightarrow \frac{1}{3}(y-1)^3 = xC_1 + C_2$$

$$\Leftrightarrow \boxed{(y-1)^3 = C_1x + C_2}$$

Let $C_1 = 0$. This gives the particular solution $y = C$

EX 5: $(4xy + 6y^2)y' + 3x^2 + 2y^2 = 0$

$$\Leftrightarrow \underbrace{(4xy + 6y^2)}_{N}dy + \underbrace{(3x^2 + 2y^2)}_{M}dx = 0$$

This equation is both homogeneous and exact, as

$$\frac{\partial M}{\partial y} = 4y = \frac{\partial N}{\partial x}$$

For this type of equation, it is usually easier to treat it as exact equation, since homogeneous equations usually require partial fraction

(EX 5) Continue

$$\phi = \int M\,dx + h(y) = x^3 + 2xy^2 + h(y)$$

$$\frac{\partial}{\partial y}\phi = 4xy + h'(y) = N = 4xy + 6y^2$$

$$\Downarrow$$

$$h'(y) = 6y^2 \Rightarrow h(y) = 2y^3$$

So $x^3 + 2xy^2 + 2y^3 = C$ is the general solution.

EX 6: $x\,dy - 4y\,dx = x\sqrt{y}\,dx$

$\Leftrightarrow y' - \frac{4}{x}y = \sqrt{y}$ Bernoulli with $n = \frac{1}{2}$

Let $u = y^{1-\frac{1}{2}} = \sqrt{y}$ and divide both sides by $\sqrt{y}$

$$u' + \frac{1}{2}\left(-\frac{4}{x}\right)u = \frac{1}{2} \Leftrightarrow u' - \frac{2}{x}u = \frac{1}{2}$$

$$I(x) = e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$$

$$\left(\frac{1}{x^2}u\right)' = \frac{1}{2x^2} \Rightarrow u = cx^2 - \frac{1}{2}x = \sqrt{y}$$

$$y = \left(cx^2 - \frac{1}{2}x\right)^2$$

EX 7: $x^2 + t^2\frac{dx}{dt} = tx\frac{dx}{dt} \Leftrightarrow \frac{dx}{dt} = \frac{x^2}{tx - t^2}$ homogeneous

Let $y = \frac{x}{t} \Rightarrow ty' + y = \frac{y^2}{y-1} \Rightarrow ty' = \frac{y}{y-1}$

$$\Rightarrow \frac{y-1}{y}\,dy = \frac{1}{t}\,dt \Rightarrow \ln|y| - y = \ln|t| + C$$

$$ye^{-y} = Ct$$